

Approximation of $\operatorname{sgn}(x)$ on two symmetric intervals by rational functions with fixed poles

Dedicated to Professor E.B. Saff on the occasion of his 70th birthday

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Abstract Recently A. Eremenko and P. Yuditskii found explicit solutions of the best polynomial approximation problems of $\operatorname{sgn}(x)$ over two intervals in terms of conformal mappings onto special comb domains. We give analogous solutions for the best approximation problems of $\operatorname{sgn}(x)$ over two symmetric intervals by odd rational functions with fixed poles. Here the existence of the related conformal mapping is proved by using convexity of the comb domains along the imaginary axis.

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1 Introduction

Conformal mappings were used in solutions of extremal problems of approximation theory from the very beginning. In fact P.L. Chebyshev gave polynomials of the least deviation from zero on $[-1, 1]$ in terms of the conformal mapping of $\mathbb{C} \setminus [-1, 1]$ onto $\{|w| < 1\}$. Later N.I. Akhiezer [1] used conformal mappings of $\mathbb{C} \setminus ([-1, a] \cup [b, 1])$ onto a rectangle and of $\mathbb{C} \setminus ([-1, a] \cup [c, d] \cup [b, 1])$ onto a fundamental domain of a Schottky group to find the least deviating polynomials on

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two intervals $[-1, a] \cup [b, 1]$. For several intervals generalizations of Akhiezer's constructions can be found in [2],[3],[8],[9],[10],[11],[12],[14],[19],[21], [22] and references therein.

Recently one of those (comb domains) was heavily used to solve several extremal problems of the approximation theory by polynomials (see, for example, survey [7]). In particular, the paper [6] contains an explicit formula for the best approximation polynomials of the function $\operatorname{sgn}(x)$ on two symmetric intervals in terms of a related comb function (the problem has many applications, see, for example [4],[20]).

The main goal of the paper is to show how more complicated comb domains can be used to solve extremal problems of the approximation theory by rational functions with fixed poles. Note also that additional difficulties appear here because of the question about the existence for desired conformal mappings.

Let $0 < x_1 < \dots < x_q < a < 1 < x_{q+1} < \dots < x_p$. For $B_0, B_1, \dots, B_p \in (0, \infty)$, $\mathbf{B} = (B_0, B_1, \dots, B_p)$, $q \leq p$, $k_0, k_1, \dots, k_p, m \in \mathbb{N}$, $\mathbf{k} = (k_0, k_1, \dots, k_p)$, denote $\Omega_{q,m,p}^{\mathbf{k}}(\mathbf{B})$ a subdomain of the half-strip

$$\left\{ w = u + iv : v > 0, \ 0 < u < \pi \left(\sum_{j=0}^p k_j + m \right) \right\}$$

which is obtained after deletion of the subset

$$E = \left\{ w = u + iv : \left| u - \pi \sum_{j=0}^q k_j \right| \leq \arccos \left(\frac{\cosh B_0}{\cosh v} \right), \ v \geq B_0 \right\}$$

and the rays

$$\begin{aligned} l_q &= \{w = u + iv : u = \pi k_q, \ v \geq B_q\}, \\ l_{q-1} &= \{w = u + iv : u = \pi(k_q + k_{q-1}), \ v \geq B_{q-1}\}, \dots, \\ l_1 &= \{w = u + iv : u = \pi(k_q + k_{q-1} + \dots + k_1), \ v \geq B_1\}, \\ l_p &= \left\{ w = u + iv : u = \pi \left(\sum_{j=0}^q k_j + m \right), \ v \geq B_p \right\}, \\ l_{p-1} &= \left\{ w = u + iv : u = \pi \left(\sum_{j=0}^q k_j + m + k_p \right), \ v \geq B_{p-1} \right\}, \dots, \\ l_{q+1} &= \left\{ w = u + iv : u = \pi \left(\sum_{j=0, j \neq q+1}^p k_j + m \right), \ v \geq B_{q+1} \right\}. \end{aligned}$$

Let $\phi(z) = \phi(z; q, m, p, \mathbf{k}, \mathbf{B})$ be a conformal map from the first quadrant $\{z = x + iy : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ onto $\Omega_{q,m,p}^{\mathbf{k}}(\mathbf{B})$ such that $\phi(a) = 0$, $\phi(1) = \pi(\sum_{j=0}^p k_j + m)$, $\phi(\infty)$ corresponds to ∞ reached along the left-hand side of l_p .

The first theorem answers the question about the existence of desired conformal mapping.

Theorem 1 *There is a unique vector $\mathbf{B}^* = \mathbf{B}^*(x_1, \dots, x_p; a)$ such that $\phi(x_j)$ correspond to ∞ reached along the left-hand side of l_j , $j = 1, \dots, q$, $\phi(0)$ corresponds to ∞ reached along the right-hand side of l_1 , $\phi(x_j)$ correspond to ∞ reached along the right-hand side of l_j , $j = q+1, \dots, p$.*

Theorem 1 conditions for conformal mapping $w = f(z)$ combine requirements of different kinds. In the preimage domain of z -plane we have fixed points corresponding to the angular image points and infinity. In the image domain of w -plane we have prescribed distances between vertical boundary rays and the analytic description of the curvilinear boundary part up to a free parameter B_0 but there is no information on ray tips.

Second theorem gives exact solution of an approximation problem in terms of the conformal mapping $\phi(z; q, m, p, \mathbf{k}, \mathbf{B}^*)$.

Theorem 2 *The error of the best uniform approximation of the function $\operatorname{sgn}(x)$, $|x| \in [a, 1]$, by rational functions of the form*

$$\frac{c_0 x^n + c_{n-1} x^{n-1} + \dots + c_0}{x^{2k_0-1}(x^2 - x_1^2)^{k_1} \dots (x^2 - x_p^2)^{k_p}}, \quad n = 2(m-1 + \sum_{j=0}^p k_j), \quad (1)$$

is equal to

$$L = \frac{1}{\cosh B_0^*(x_1, \dots, x_p; a)}, \quad (2)$$

and the extremal rational function is given by

$$f(x) = 1 - (-1)^{\sum_{j=0}^q k_j} L \cos \phi(x; q, m, p, \mathbf{k}; \mathbf{B}^*), \quad x > 0.$$

Other forms of approximating rational functions are considered analogously. For example, in the case of an even order pole at the origin instead of the left part of the boundary of the set E the vertical half-line

$$\left\{ w = u + iv : u = \pi \sum_{j=0}^q k_j, v \geq B_0 \right\},$$

should be considered. Note also that for $p = 0$ Theorem 2 was proved in [15].

2 Proofs

Proof of Theorem 1

First, map the upper half-disk $\{\zeta : |\zeta| < 1, \operatorname{Im} \zeta > 0\}$ onto the first quadrant by a fixed function $z = z(\zeta)$ so that $z(-1) = a$, $z(1) = 1$ and $z(i) = 0$. Denote $z^{-1}(x_j) := e^{i\alpha_j}$, $j = 1, \dots, p$; $z^{-1}(\infty) := e^{i\alpha_{p+1}}$, $0 < \alpha_{q+1} < \dots < \alpha_p < \alpha_{p+1} < \alpha_0 < \alpha_1 < \dots < \alpha_q < \pi$, where $\alpha_0 := \frac{\pi}{2}$. Then $f(\zeta) := \phi(z(\zeta))$ maps the upper half-disk onto $\Omega_{q,m,p}^{\mathbf{k}}(\mathbf{B})$ so that $f(-1) = 0$, $f(1) = \pi(\sum_{j=0}^p k_j + m)$, $f(e^{i\alpha_0})$ corresponds to ∞ reached along the right-hand side of l_1 , $f(e^{i\alpha_{p+1}})$ corresponds to ∞ reached along the left-hand side of l_p , $f(e^{i\alpha_j})$

correspond to ∞ reached along the left-hand side of l_j , $j = 1, \dots, q$; $f(e^{i\alpha_j})$ correspond to ∞ reached along the right-hand side of l_j , $j = q+1, \dots, p$.

The problem admits an equivalent posing for $f(\zeta)$ with the boundary conditions being transformed from $\phi(z)$ to $f(\zeta)$.

Second, extend f continuously onto the boundary of the upper half-disk, and apply the Schwarz symmetry principle to extend f onto the lower half-disk $\{\zeta : |\zeta| < 1, \operatorname{Im} \zeta < 0\}$. The extended function denoted also by f maps the unit disk \mathbb{D} conformally onto the domain $\Omega := \Omega_{q,m,p}^{\mathbf{k}}(\mathbf{B}) \cup (0, \pi(\sum_{j=0}^p k_j + m)) \cup \overline{\Omega_{q,m,p}^{\mathbf{k}}(\mathbf{B})}$, where $\overline{\Omega_{q,m,p}^{\mathbf{k}}(\mathbf{B})}$ is a reflection of $\Omega_{q,m,p}^{\mathbf{k}}(\mathbf{B})$ into the lower half-plane.

The domain Ω is symmetric with respect to the real axis and is convex in the direction of imaginary axis which means that an intersection of every vertical straight line with Ω is either empty or connected. We say that a function g is convex in the direction of imaginary axis if it maps \mathbb{D} onto a domain convex in the direction of imaginary axis. The integral representation of functions convex in the direction of imaginary axis is known, see, e.g., [17], [16]. Namely, such function g is given by the formula

$$g(\zeta) = C_1 \int_0^\zeta \frac{h(t)dt}{(1 - \bar{\zeta}_1 t)(1 - \bar{\zeta}_2 t)} + C_2, \quad (3)$$

with constant real numbers C_1 and C_2 , and a function h , $h(0) = 1$, which is holomorphic in \mathbb{D} and satisfies the condition $\operatorname{Re}(e^{i\gamma} h(\zeta)) > 0$, $\zeta \in \mathbb{D}$, γ is fixed, $\gamma \in (-\pi/2, \pi/2)$, while the points ζ_1, ζ_2 , $|\zeta_1| = |\zeta_2| = 1$, are disposed so that all boundary points of the image of \mathbb{D} under the mapping $w = \zeta[(1 - \bar{\zeta}_1 \zeta)(1 - \bar{\zeta}_2 \zeta)]^{-1}$ belong to the straight line $\{w : \operatorname{Re}(e^{i\gamma} w) = 0\}$. Since the constructed function f is convex in the direction of imaginary axis and f has real Taylor coefficients, we conclude that f is represented by integral (3) with $\gamma = 0$, $\zeta_1 = -1$, $\zeta_2 = 1$. From the other side, every function (3) is convex in the direction of imaginary axis, and it has real Taylor coefficients if $\gamma = 0$, $\zeta_1 = -1$, $\zeta_2 = 1$.

The aim of the proof is to show that a function h in representation (3) for f is uniquely determined by the boundary conditions.

The function h in (3) is represented by the Herglotz integral, see, e.g., [5, p.22],

$$h(\zeta) = \int_{-\pi}^{\pi} \frac{1 + \zeta e^{-i\varphi}}{1 - \zeta e^{-i\varphi}} d\mu(\varphi), \quad |\zeta| < 1, \quad (4)$$

where $d\mu$ is a positive measure, $\mu(\pi) - \mu(-\pi) = 1$. The geometry of the domain Ω determines a special behavior of μ . Denote $I := (\alpha_{p+1}, \alpha_0)$ and $-I := (-\alpha_0, -\alpha_{p+1})$. The function μ has to be piecewise constant on $[-\pi, \pi] \setminus ((-I) \cup I)$ with jumps at $\varphi = -\alpha_j$ and $\varphi = \alpha_j$, $j = 0, 1, \dots, p+1$. Therefore,

$$h(\zeta) = \sum_{j=0}^{p+1} \lambda_j \left(\frac{1 + \zeta e^{i\alpha_j}}{1 - \zeta e^{i\alpha_j}} + \frac{1 + \zeta e^{-i\alpha_j}}{1 - \zeta e^{-i\alpha_j}} \right) + \int_{(-I) \cup I} \frac{1 + \zeta e^{-i\varphi}}{1 - \zeta e^{-i\varphi}} d\mu(\varphi), \quad |\zeta| < 1. \quad (5)$$

Here $\mu(\varphi)$ is continuous on $(-I) \cup I$, $d\mu(\varphi) = d\mu(-\varphi)$, $\lambda_j > 0$, $j = 0, 1, \dots, p+1$, and

$$2 \sum_{j=0}^{p+1} \lambda_j + 2 \int_I d\mu(\varphi) = 1. \quad (6)$$

Positive numbers $\lambda_0, \lambda_1, \dots, \lambda_{p+1}, C_1$ and C_2 and the measure $d\mu$ on I and $(-I)$ in (4) and (5) are the accessory parameters of the problem. We will show that a function (3) with a suitable choice of the accessory parameters will coincide with the needed function $f(\zeta) = \phi(z(\zeta))$.

Choose C_2 to satisfy the condition $f(-1) = 0$ which, according to (3) gives the representation

$$f(\zeta) = C_1 \int_{-1}^{\zeta} \frac{h(t)dt}{(1 - \bar{\zeta}_1 t)(1 - \bar{\zeta}_2 t)}. \quad (7)$$

The function f in representations (7) and (5) has singular points at $e^{i\alpha_0}, e^{i\alpha_1}, \dots, e^{i\alpha_p}, e^{i\alpha_{p+1}}$, and $e^{-i\alpha_0}, e^{-i\alpha_1}, \dots, e^{-i\alpha_p}, e^{-i\alpha_{p+1}}$. Note that representation (7) does not imply singularities at $\zeta = -1$ and $\zeta = 1$ because $h(-1) = h(1) = 0$.

Examine the character of singularity at $\zeta = e^{i\alpha_q}$. In a neighborhood of $\zeta = e^{i\alpha_q}$ the function (7) has an expansion

$$f(\zeta) = \frac{C_1 \lambda_q}{i \sin \alpha_q} \log(1 - \zeta e^{-i\alpha_q}) + H_q(\zeta), \quad (8)$$

where H_q is holomorphic in a neighborhood of $\zeta = e^{i\alpha_q}$. The argument of $(1 - \zeta e^{-i\alpha_q})$ has a jump equal to π when ζ moves along the unit circle through the point $e^{i\alpha_q}$.

Thus, as ζ moves along the unit circle through $e^{i\alpha_q}$, $\operatorname{Re} f(\zeta)$ has a jump d_q ,

$$d_q = \frac{C_1 \lambda_q \pi}{\sin \alpha_q}. \quad (9)$$

Evidently, d_q is equal to a distance between the imaginary axis and a vertical ray which is the image of the arc $\{\zeta = e^{i\theta} : \alpha_{q-1} < \theta < \alpha_q\}$ under $w = f(\zeta)$.

Similarly, in a neighborhood of $\zeta = e^{i\alpha_{q-1}}$, $f(\zeta)$ has an expansion

$$f(\zeta) = \frac{C_1 \lambda_{q-1}}{i \sin \alpha_{q-1}} \log(1 - \zeta e^{-i\alpha_{q-1}}) + H_{q-1}(\zeta), \quad (10)$$

where H_{q-1} is holomorphic in a neighborhood of $\zeta = e^{i\alpha_{q-1}}$. As ζ moves along the unit circle through $e^{i\alpha_{q-1}}$, $\operatorname{Re} f(\zeta)$ has a jump d_{q-1} ,

$$d_{q-1} = \frac{C_1 \lambda_{q-1} \pi}{\sin \alpha_{q-1}}, \quad (11)$$

where d_{q-1} is equal to a distance between the vertical rays which are the images of the arcs $\{\zeta = e^{i\theta} : \alpha_{q-1} < \theta < \alpha_q\}$ and $\{\zeta = e^{i\theta} : \alpha_{q-2} < \theta < \alpha_{q-1}\}$ under $w = f(\zeta)$.

As for the domain Ω , the distance between the imaginary axis and l_q in Ω equals πk_q , and the distance between l_q and l_{q-1} in Ω equals πk_{q-1} . In order to preserve a proportion between the distances in Ω and the corresponding distances in $f(\mathbb{D})$, according to (9) and (11), we have

$$\lambda_q = \lambda_{q-1} \frac{k_q}{k_{q-1}} \frac{\sin \alpha_q}{\sin \alpha_{q-1}}. \quad (12)$$

In the same way,

$$\lambda_{q-1} = \lambda_{q-2} \frac{k_{q-1}}{k_{q-2}} \frac{\sin \alpha_{q-1}}{\sin \alpha_{q-2}}, \dots, \lambda_2 = \lambda_1 \frac{k_2}{k_1} \frac{\sin \alpha_2}{\sin \alpha_1},$$

which implies that

$$\lambda_j = \lambda_1 \frac{k_j}{k_1} \frac{\sin \alpha_j}{\sin \alpha_1}, \dots, j = 2, \dots, q. \quad (13)$$

Besides, since a distance between l_1 and the curvilinear boundary of Ω equals $\pi(k_0 - \frac{1}{2})$, we have

$$\lambda_0 = \lambda_1 \frac{k_0 - \frac{1}{2}}{k_1} \frac{\sin \alpha_0}{\sin \alpha_1}. \quad (14)$$

From the other side,

$$\lambda_{q+1} = \lambda_{q+2} \frac{k_{q+1}}{k_{q+2}} \frac{\sin \alpha_{q+1}}{\sin \alpha_{q+2}}, \dots, \lambda_{p-1} = \lambda_p \frac{k_{p-1}}{k_p} \frac{\sin \alpha_{p-1}}{\sin \alpha_p},$$

which implies that

$$\lambda_j = \lambda_p \frac{k_j}{k_p} \frac{\sin \alpha_j}{\sin \alpha_p}, j = q+1, \dots, p-1. \quad (15)$$

Besides, since a distance between l_p and the curvilinear boundary of Ω equals $\pi(m - \frac{1}{2})$, we have

$$\lambda_{p+1} = \lambda_p \frac{m - \frac{1}{2}}{k_p} \frac{\sin \alpha_{p+1}}{\sin \alpha_p}. \quad (16)$$

To compare λ_1 and λ_p , write one more relation

$$\lambda_p = \lambda_1 \frac{k_p}{k_1} \frac{\sin \alpha_p}{\sin \alpha_1}. \quad (17)$$

Take into account (13) - (17) and observe that there remain three undetermined accessory parameters C_1 , λ_1 and $d\mu$ on I and $(-I)$ subject to relation (6). Choose the scaling parameter C_1 so that the distance between the images of $\{\zeta = e^{i\theta} : \alpha_1 < \theta < \alpha_2\}$ and $\{\zeta = e^{i\theta} : \alpha_0 < \theta < \alpha_1\}$ under $w = f(\zeta)$ has to be equal to the distance between l_1 and l_2 ,

$$C_1 = \frac{k_1 \sin \alpha_1}{\lambda_1}. \quad (18)$$

The positive measure $d\mu$ on $(-I) \cup I$ is of more complicated nature than the remaining number accessory parameter $\lambda_1 \in (0, 1/2)$. The variation $\mu(\alpha_0 - 0) - \mu(\alpha_{p+1} + 0)$ depends on λ_1 by (6),

$$\operatorname{var}_I \mu := \int_I d\mu(\varphi) = \frac{1}{2} - \sum_{j=0}^{p+1} \lambda_j,$$

where $\lambda_0, \lambda_2, \dots, \lambda_{p+1}$ are linear functions of λ_1 .

Define $d\mu$ on $(-I) \cup I$. The integral in the right-hand side of (5) can be considered as the Schwarz integral with a smooth density which is equal to the real part of the mapping from $(-I) \cup I$. Require that $\mu(\varphi)$ has a continuous derivative $\mu'(\varphi)$ on $(-I) \cup I$ that represents the density of the Schwarz integral. So

$$\int_{(-I) \cup I} \frac{1 + \zeta e^{-i\varphi}}{1 - \zeta e^{-i\varphi}} d\mu(\varphi) = \int_{(-I) \cup I} \mu'(\varphi) \frac{1 + \zeta e^{-i\varphi}}{1 - \zeta e^{-i\varphi}} d\varphi, \quad |\zeta| < 1,$$

and

$$2\pi\mu'(\varphi) = \operatorname{Re} h(e^{i\varphi}), \quad \varphi \in I. \quad (19)$$

Let $\varphi \in I$ and

$$f(e^{i\varphi}) = u(\varphi) + iv(\varphi)$$

be given by (7). The value $u(\alpha_{p+1}) - u(\alpha_0)$ determines the length of a projection of $f(I)$ on the real axis. Require that this length is equal to π , i.e.,

$$\begin{aligned} u(\alpha_{p+1}) - u(\alpha_0) &= C_1 \operatorname{Re} \int_{e^{i\alpha_0}}^{e^{i\alpha_{p+1}}} \frac{h(t)dt}{1-t^2} = C_1 \operatorname{Re} \int_{\alpha_0}^{\alpha_{p+1}} \frac{h(e^{i\varphi})ie^{i\varphi}d\varphi}{1-e^{i2\varphi}} = \\ &= \frac{C_1}{2} \int_{\alpha_{p+1}}^{\alpha_0} \operatorname{Re} \frac{h(e^{i\varphi})d\varphi}{\sin \varphi} = C_1 \pi \int_{\alpha_{p+1}}^{\alpha_0} \frac{\mu'(\varphi)d\varphi}{\sin \varphi} = \pi. \end{aligned}$$

So consider $d\mu$ satisfying the restriction

$$C_1 \int_{\alpha_{p+1}}^{\alpha_0} \frac{d\mu(\varphi)}{\sin \varphi} = 1. \quad (20)$$

Evaluate $u(\alpha)$, $\alpha \in I$,

$$u(\alpha) = C_1 \operatorname{Re} \int_{-1}^{e^{i\alpha}} \frac{h(t)dt}{1-t^2} = C_1 \operatorname{Re} \left[\int_{-1}^{e^{i\alpha_0}} \frac{h(t)dt}{1-t^2} + \int_{e^{i\alpha_0}}^{e^{i\alpha}} \frac{h(t)dt}{1-t^2} \right] = \quad (21)$$

$$\pi \left(\sum_{j=0}^q k_j - \frac{1}{2} \right) + C_1 \pi \int_{\alpha}^{\alpha_0} \frac{\mu'(\varphi)d\varphi}{\sin \varphi}.$$

Now evaluate $v(\alpha)$, $\alpha \in I$,

$$v(\alpha) = C_1 \operatorname{Im} \int_{-1}^{e^{i\alpha}} \frac{h(t)dt}{1-t^2} = C_1 \operatorname{Im} \left[\int_{-1}^0 \frac{h(t)dt}{1-t^2} + \int_0^{e^{i\alpha}} \frac{h(t)dt}{1-t^2} \right] = \quad (22)$$

$$C_1 \operatorname{Im} \int_0^{e^{i\alpha}} \frac{h(t)dt}{1-t^2} = C_1 \left[\sum_{j=0}^{p+1} \frac{\lambda_j}{\sin \alpha_j} \log \frac{\sin \frac{\alpha+\alpha_j}{2}}{\sin \frac{|\alpha-\alpha_j|}{2}} + \int_{\alpha_{p+1}}^{\alpha_0} \frac{\mu'(\varphi)}{\sin \varphi} \log \frac{\sin \frac{\alpha+\varphi}{2}}{\sin \frac{|\alpha-\varphi|}{2}} d\varphi \right].$$

The curvilinear part of $\partial\Omega$ consists of the curve L in the upper half-plane and its reflection \overline{L} . The function μ' , if it exists, provides the needed boundary behavior of function f on $(-I) \cup I$ so that f maps $(-I) \cup I$ onto $L \cup \overline{L}$. This is equivalent to the relation between $u(\alpha)$ and $v(\alpha)$, $\alpha \in I$, in the form of the equation

$$v(\alpha) = \operatorname{arccosh} \left(\frac{\cosh B_0}{\cos(u(\alpha) - \pi \sum_{j=0}^q k_j)} \right), \quad \alpha \in I, \quad (23)$$

for a suitable number $B_0 > 0$. Equation (23) is the explicit representation of L . Take into account (21) and (22) to rewrite (23) in the form

$$C_1 \left[\sum_{j=0}^{p+1} \frac{\lambda_j}{\sin \alpha_j} \log \frac{\sin \frac{\alpha+\alpha_j}{2}}{\sin \frac{|\alpha-\alpha_j|}{2}} + \int_{\alpha_{p+1}}^{\alpha_0} \frac{\mu'(\varphi)}{\sin \varphi} \log \frac{\sin \frac{\alpha+\varphi}{2}}{\sin \frac{|\alpha-\varphi|}{2}} d\varphi \right] = \quad (24)$$

$$\operatorname{arccosh} \left(\frac{\cosh B_0}{\sin(C_1 \pi \int_{\alpha}^{\alpha_0} \frac{\mu'(\varphi)}{\sin \varphi} d\varphi)} \right), \quad \alpha \in I,$$

subject to restrictions (6) and (20).

Show that restrictions (6) and (20) give a non-degenerate interval for values of λ_1 . Note that, due to (13) - (17),

$$\sum_{j=0}^{p+1} \lambda_j = \lambda_1 A, \quad (25)$$

where

$$A = \frac{1}{k_1 \sin \alpha_1} \left[\sum_{j=1}^p k_j \sin \alpha_j + k_0 - \frac{1}{2} + \left(m - \frac{1}{2} \right) \sin \alpha_{p+1} \right]. \quad (26)$$

The restrictions (6) and (20) imply that

$$\frac{k_1 \sin \alpha_1}{2(1 + A k_1 \sin \alpha_1)} < \lambda_1 < \frac{k_1 \sin \alpha_1}{2(\sin \alpha_{p+1} + A k_1 \sin \alpha_1)}, \quad A > 1.$$

Equation (24) is an integral nonlinear equation with respect to $\mu'(\varphi)$, $\varphi \in I$. The value B_0 suits restriction (20). For a given μ' , relation (6) determines the value of the accessory parameter λ_1 .

We have to prove now that there exists an integrable solution μ' to equation (24). We will approximate μ on I by piecewise constant functions. To this purpose, for every natural $n \geq 1$, consider a partition $\{\beta_{1,n}, \dots, \beta_{n,n}\}$ of I ,

$$\beta_{j,n} = \alpha_{p+1} + \frac{\alpha_0 - \alpha_{p+1}}{n+1}j, \quad j = 1, \dots, n,$$

$$\Delta_{j,n} = (\beta_{j-1,n}, \beta_{j,n}), \quad j = 1, \dots, n+1, \quad \beta_{0,n} := \alpha_{p+1}, \quad \beta_{n+1,n} := \alpha_0.$$

Denote by $\mu^{(n)}$ a piecewise constant function on I which has n discontinuity points at $\beta_{1,n}, \dots, \beta_{n,n}$ and $\mu = \mu^{(n)} + \sum_{j=q+1}^{p+1} \lambda_j$ in a neighborhood of α_{p+1} . Let us construct an algorithm to define jump values $\mu_{k,n} > 0$ of $\mu^{(n)}$ at the points $\beta_{k,n}$, $k = 1, \dots, n$,

$$C_1 \sum_{k=1}^n \frac{\mu_{k,n}}{\sin \beta_{k,n}} = 1.$$

According to (5) with $d\mu$ substituted by $d\mu^{(n)}$, the measure $d\mu^{(n)}$ generates a function h_n . Let f_n be defined by (7) with h substituted by h_n . Then f_n maps \mathbb{D} onto a domain Ω_n which can be described geometrically in the following way. The domain Ω includes the part $E \cup \overline{E}$ of the vertical strip S of width π bounded by L and \overline{L} , where \overline{E} is a reflection of E . Substitute $E \cup \overline{E}$ by the strip S slit along n vertical rays $\mathcal{L}_{1,n}, \dots, \mathcal{L}_{n,n}$ in the upper half-plane and their reflections and obtain Ω_n .

Denote by $w_{1,n}, \dots, w_{n,n}$ the tips of rays $\mathcal{L}_{1,n}, \dots, \mathcal{L}_{n,n}$, respectively,

$$\operatorname{Re} w_{k,n} = \pi \left(\sum_{j=0}^q k_j + \frac{1}{2} \right) - \sum_{j=1}^k \frac{C_1 \pi \mu_{j,n}}{\sin \beta_{j,n}}, \quad k = 1, \dots, n.$$

The points $w_{k,n}$ are images of $e^{i\varphi_{k,n}}$, $k = 1, \dots, n$, under the map f_n . All values $\varphi_{1,n}, \dots, \varphi_{n,n}$ depend on $\mu_{1,n}, \dots, \mu_{n,n}$. It follows from results of [18] up to the Möbius transformation that, for every $k = 1, \dots, n$, $\varphi_{k,n}(\mu_{1,n}, \dots, \mu_{n,n})$ tends asymptotically to the center point of $\Delta_{k,n}$ as $\mu_{k,n} \rightarrow 0$.

The function $u_n(\varphi) = \operatorname{Re} f_n(e^{i\varphi})$, $\varphi \in I$, is piecewise constant,

$$u_n(\varphi) = \operatorname{Re} w_{k,n}, \quad \varphi \in \Delta_{k,n}, \quad k = 1, \dots, n.$$

The function $v_n(\varphi) = \operatorname{Im} f_n(e^{i\varphi})$ is continuous on all $\Delta_{k,n}$, $k = 1, \dots, n$,

$$v_n(\varphi) = C_1 \left[\sum_{j=0}^{p+1} \frac{\lambda_j}{\sin \alpha_j} \log \frac{\sin \frac{\varphi + \alpha_j}{2}}{\sin \frac{|\varphi - \alpha_j|}{2}} + \sum_{j=1}^n \frac{\mu_{j,n}}{\sin \beta_{j,n}} \log \frac{\sin \frac{\varphi + \beta_{j,n}}{2}}{\sin \frac{|\varphi - \beta_{j,n}|}{2}} \right]$$

for $\varphi \in \Delta_{k,n}$, $k = 1, \dots, n$,

$$v_n(\varphi_{k,n}) = \operatorname{Im} w_{k,n} = \min_{\varphi \in \Delta_{k,n}} v_n(\varphi), \quad k = 1, \dots, n.$$

We have n free parameters $\mu_{1,n}, \dots, \mu_{n,n}$ subject to restriction (20) and consider B_0 as one more free parameter $B_{0,n}$ to establish n values $v_n(\varphi_{k,n})$,

$k = 1, \dots, n$, satisfying (23). If $B_{0,n}$ tends to 0, then Ω_n degenerates. If $B_{0,n}$ tends to infinity, then E degenerates. Values $v_n(\varphi_{k,n})$ are strictly monotone in $\mu_{j,n}$, $j, k = 1, \dots, n$, and $B_{0,n}$.

For $n > 1$, call $(\mu_{1,n}, \dots, \mu_{n,n})$ *admissible* if

$$C_1 \sum_{k=1}^{n-1} \frac{\mu_{k,n}}{\sin \beta_{k,n}} < 1.$$

Vary $B_{0,n}$ to find $\mu_{n,n}$ for which restriction (20) and condition (23) are satisfied at $\alpha = \varphi_{n,n}$ with $u = u_n(\varphi_{n,n}) = \operatorname{Re} w_{n,n}$ and $v = v_n(\varphi_{n,n}) = \operatorname{Im} w_{n,n}$. In this way, we find $\mu_{n,n} = \mu_{n,n}(\mu_{1,n}, \dots, \mu_{n-1,n})$ and $B_{0,n} = B_{0,n}(\mu_{1,n}, \dots, \mu_{n-1,n})$ for any admissible $(\mu_{1,n}, \dots, \mu_{n-1,n})$.

Next, for $n > 2$ and admissible $(\mu_{1,n}, \dots, \mu_{n-2,n})$ such that

$$C_1 \sum_{k=1}^{n-2} \frac{\mu_{k,n}}{\sin \beta_{k,n}} < 1,$$

find $\mu_{n-1,n} = \mu_{n-1,n}(\mu_{1,n}, \dots, \mu_{n-2,n})$ for which (23) is satisfied at $\alpha = \varphi_{n-1,n}$ with $u = u_n(\varphi_{n-1,n}) = \operatorname{Re} w_{n-1,n}$ and $v = v_n(\varphi_{n-1,n}) = \operatorname{Im} w_{n-1,n}$ where $\mu_{n,n} = \mu_{n,n}(\mu_{1,n}, \dots, \mu_{n-1,n})$ and $B_{0,n} = B_{0,n}(\mu_{1,n}, \dots, \mu_{n-1,n})$. A solution exists since $\mu_{n-1,n}$ depends continuously on $\mu_{1,n}, \dots, \mu_{n-2,n}$ and the two extreme positions of admissible $\mu_{n-1,n}$ lead to degenerate configurations.

Continue this process up to the last step and find all $\mu_{1,n}, \dots, \mu_{n,n}$ and $B_{0,n}$ solving (23) at every $\alpha = \varphi_{k,n}$, $k = 1, \dots, n$, with $u = u_n$ and $v = v_n$.

When n tends to infinity, Helly's theorems admit existence of a convergent subsequence from $\{\mu^{(n)}\}$ for which $B_{0,n}$ tends to a certain B_0 . The corresponding subsequence of functions h_n generated by $\{d\mu^{(n)}\}$ converges to a function h and the subsequence $\{f_n\}$ given by (7) with h_n in its right-hand side converges to f^* . Preserve the denotation $\mu^{(n)}$ for the convergent subsequence. The variation for the limit measure satisfies (20). The limit function f^* is not constant because of the normalization condition (20). Therefore, f^* maps \mathbb{D} onto a kernel Ω^* of the subsequence of domains $\{\Omega_n\}$. Evidently, $\Omega^* = \Omega$ if the rays $\{\mathcal{L}_{k,n}\}$, $1 \leq k \leq n$, $n \geq 1$, are dense in the strip S . This is equivalent to the condition

$$\lim_{n \rightarrow \infty} \max_{1 \leq k \leq n} \mu_{k,n} = 0.$$

Suppose that there exists a sequence $\{k_n\}$, $1 \leq k \leq n$, (for a suitable subsequence of n 's) such that

$$\lim_{n \rightarrow \infty} \mu_{k_n,n} = \mu_0 > 0.$$

This means that $\partial\Omega^*$ contains a vertical straight segment \mathcal{L}_0 in the closure of the strip S . A preimage of \mathcal{L}_0 under f^* is an arc in $\partial\mathbb{D}$. Let $e^{i\varphi^*}$ be an inner point of this arc, $\varphi^* \in I$. In a neighborhood of φ^* there is a set of $\Delta_{k,n}$ with n large enough and certain integers k , $w_{k,n} \in \Delta_{k,n}$. By construction,

$$\frac{\operatorname{Im}(w_{k,n} - w_{j,n})}{\operatorname{Re}(w_{k,n} - w_{j,n})}$$

is bounded. To the contrary, this ratio is unbounded if $\Delta_{k,n}$ and $\Delta_{j,n}$ are in a neighborhood of φ^* . The contradiction implies that the supposition is wrong, and the rays $\{\mathcal{L}_{k,n}\}$, $1 \leq k \leq n$, $n \geq 1$, are dense in S .

Finally, we obtained a smooth measure $d\mu$ which is the limit of subsequence of $\{d\mu^{(n)}\}$. Equation (6) with this $d\mu$ gives us $\sum_{j=0}^{p+1} \lambda_j$. Hence, according to (25) and (26), we determine the last free parameter λ_1 and complete the proof.

Proof of Theorem 2. Proof is quite similar to the proof of [6, Theorem 4] (compare also [13, Theorem 3.1]). First of all we note that

$$f = 1 - (-1)^{\sum_{j=0}^q k_j} L \cos \phi$$

is real on the positive real semi-axis and pure imaginary on the positive imaginary semi-axis. So by two reflections with respect to the coordinate axes, f extends to an odd function analytic in $\mathbb{C} \setminus \{0, \pm x_1, \dots, \pm x_p\}$. The region $\Omega_{q,m,p}^k(\mathbf{B}^*)$ is close to the strip

$$\left\{ w : \operatorname{Re} w \in \left(\pi \sum_{j=1}^q k_j, \pi \left(\sum_{j=0}^q k_j - 1/2 \right) \right) \right\}$$

as $\operatorname{Im} w \rightarrow \infty$ reached along the right-hand side of l_1 , and to the strip

$$\left\{ w : \operatorname{Re} w \in \left(\pi \left(\sum_{j=0}^q k_j + 1/2 \right), \pi \left(\sum_{j=0}^q k_j + m \right) \right) \right\}$$

as $\operatorname{Im} w \rightarrow \infty$ reached along the left-hand side of l_p . So $\phi \sim (2k_0 - 1) \log 1/z$, $z \rightarrow 0$, $\phi \sim (2m - 1) \log z$, $z \rightarrow \infty$, $\phi \sim k_j \log(1/(z \pm x_j))$, $z \rightarrow \pm x_j$, $j = 1, \dots, p$, and, therefore f is a rational function of the form (1). Next we note that the graph of f alternates $\sum_{j=0}^p k_j + m + 1$ times on $[a, 1]$ between $1 - L$ and $1 + L$. Now we conclude the proof by using the general Chebyshev alternation theorem on the uniform approximation of continuous functions by Chebyshev systems (here we consider the Chebyshev system

$$\frac{1}{x^{2k_0-1}(x^2 - x_1^2)^{k_1} \dots (x^2 - x_p^2)^{k_p}}, \dots, \frac{x^n}{x^{2k_0-1}(x^2 - x_1^2)^{k_1} \dots (x^2 - x_p^2)^{k_p}}$$

on $[-1, a] \cup [a, 1]$.)

Finally we note there is a unique point on the imaginary axis where $\phi = \pi \sum_{j=0}^q k_j + iB_0^*$, what gives (2).

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